

Morita's duality for split reductive groups

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ABSTRACT. In this paper, we extend the work in *Morita's Theory for the Symplectic Groups* [7] to split reductive groups. We construct and study the holomorphic discrete series representation and the principal series representation of a split reductive group G over a p -adic field F as well as a duality between certain sub-representations of these two representations.

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Notations

Let p be a prime, F a finite extension of \mathbb{Q}_p , \mathfrak{o} the ring of integers of F , ϖ a uniformizer of \mathfrak{o} , $|\cdot|$ the normalized absolute value, and F^{alg} an algebraic closure of F . Let K be an extension of F with an absolute value extending $|\cdot|$, and \mathfrak{o}_K the valuation ring of K . We assume that K is complete with respect to $|\cdot|$, and moreover, K is spherically complete whenever topological properties of the K -vector spaces are under consideration.

1. Introduction

In a series of papers, [4], [5] and [6], Morita and Murase initiated the work on the representation theory for $\text{SL}(2, F)$ with coefficient field K , especially holomorphic discrete

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series representations and their duality relations with principal series representations. Principal series representations, or more generally, induced representations, appeared in many literatures, notably in Féaux de Lacroix's work [2] on locally analytic representations. On the other hand, holomorphic discrete series representations were not extensively studied. The holomorphic discrete series representation of $\mathrm{SL}(n+1, F)$ associated to a rational representation of $\mathrm{GL}(n, F)$ were introduced by Schneider in [8], in order to understand the de Rham complex over Drinfel'd's space as representation of $\mathrm{SL}(n+1, F)$. In another direction, the holomorphic discrete series representation of $\mathrm{Sp}(2n, F)$ associated to a K -rational representation of $\mathrm{GL}(n, F)$ were constructed in our recent work [7]. Furthermore, the algebraization and generalization of Morita's duality were established in [7].

The purpose of this paper is to generalize Morita's theory from $\mathrm{Sp}(2n, F)$ to a split reductive group G . We are able to do such a generalization owing to the entirely algebraic construction of Morita's theory for $\mathrm{Sp}(2n, F)$. Therefore, we shall closely follow the main ideas presented in [7].

In the first paragraph, we recollect some notions on split reductive groups and construct an F -regular function f on G that characterizes the parabolic big cell associated to a parabolic subgroup. In particular, f corresponds to the determinant function on $M(n, F)$ used in the definition of p -adic Siegel upper half-space in [7] (see Example 2.5 and 4.3). f will appear extensively in the study of the rigid symmetric space associated to G and holomorphic series representations.

The principal series representation $(C_\sigma^{\mathrm{an}}(\mathfrak{H}, V), T_\sigma)$ is another interpretation of the parabolic induction from a locally analytic K -representation (σ, V) of the Levi subgroup (cf. [5] and [7]). In the second paragraph, applying the general results of Féaux de Lacroix on induced representations of F -Lie groups ([2]), one sees that $(C_\sigma^{\mathrm{an}}(\mathfrak{H}, V), T_\sigma)$ is a locally analytic representation of G over a K -vector space of compact type.

The third paragraph is dedicated to the construction and study of the rigid analytic symmetric space Ω , which is the foundation of holomorphic discrete series representations. Some examples are the p -adic upper half-plane for $\mathrm{SL}(2, F)$ (cf. [4]), Drinfel'd's space for $\mathrm{SL}(n+1, F)$ (cf. [8]) and the p -adic Siegel upper half-space for $\mathrm{Sp}(2n, F)$ (cf. [7]). Such symmetric spaces have been studied by van der Put and Voskuil in [14]. We shall however use another approach following [10] and [7] to construct the admissible affinoid covering, which enables us to obtain precise descriptions of rigid analytic functions on Ω . From this, we prove that the space $\mathcal{O}_K(\Omega)$ of K -rigid analytic functions on Ω_K is a nuclear K -Fréchet space.

In the fourth paragraph, for a K -rational representation (σ, V) of the Levi subgroup, we construct the holomorphic discrete series representation $(\mathcal{O}_\sigma(\Omega), \pi_\sigma)$ defined over the nuclear K -Fréchet space of V -valued rigid analytic functions on Ω . Moreover, we prove that its dual representation is locally analytic.

Since the strong duality gives rise to a contra-variant equivalence between the category of K -vector spaces of compact type and the category of nuclear K -Fréchet spaces (cf.

[11]), it is natural to expect certain duality relations between sub-quotient spaces of principal series representations and those of holomorphic discrete series representations. For $\mathrm{SL}(2, F)$, a duality of this kind via residues is analytically constructed by Morita (cf. [5]). However, there does not seem to be any direct way to generalize Morita's duality. Nevertheless, Morita's duality may be algebraically interpreted in a weaker form and generalized to any split reductive group G . These are done in the last paragraph. For a K -rational representation (σ, V) of the Levi subgroup, a closed sub-representation $B_{\sigma^*}(\mathfrak{H}, V^*)$ of $C_{\sigma^*}^{\mathrm{an}}(\mathfrak{H}, V^*)$ and a closed sub-representation $\mathcal{N}_{\sigma}(\Omega)$ of $\mathcal{O}_{\sigma}(\Omega)$ are algebraically constructed, along with a duality between them. As discussed in [7], our duality for $\mathrm{SL}(2, F)$ is exactly Morita's duality composed with Casselman's intertwining operator (cf. [5]).

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2. A lemma on the split reductive groups

2.1. Split reductive groups. We adopt the notations in [3] Part II, Chapter 1.

Let G be a connected split reductive algebraic group over F , T a split maximal torus of G . We have the decomposition of Lie algebra \mathfrak{g} of G (over F) in the form

$$(2.1) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha},$$

where \mathfrak{g}_0 is the Lie algebra of T and R is the root system of G with respect to T .

Each \mathfrak{g}_{α} is of rank 1 over F , and we denote $U_{\alpha} \simeq \mathbb{G}_a(F)$ the root subgroup of G corresponding to α .

Let $W \cong N_G(T)/T$ be the Weyl group of R . For $w \in W$, we also denote w a representing element in $N_G(T)$.

Choose a positive system R^+ and denote S the corresponding set of simple roots. Let B^+ denote the corresponding Borel subgroup and B^- its opposite Borel subgroup, $U_G^{\pm} = U(\pm R^+)$ the unipotent radical of B^{\pm} .

Throughout this article we fix a subset I of S , and denote $R_I = R \cap \mathbb{Z}I$, W_I the Weyl group of R_I , $R_I^+ = R^+ - R_I$, P^+ the standard parabolic subgroup of G corresponding to R_I^+ , P^- its opposite parabolic subgroup, $U^{\pm} = U(\pm R_I^+)$ the unipotent radical of P^{\pm} , L the common Levi subgroup of P^+ and P^- .

L is a split reductive group with split maximal torus T , root system R_I , positive system $R_I^+ = R_I \cap R^+$ and Weyl group W_I . Let $U_L^{\pm} = U(\pm R_I^+)$.

We recall ([3] Part II.1.7) that for any closed and unipotent subset R' of R (that is, $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap R \subset R'$ for any $\alpha, \beta \in R'$ and $R' \cap (-R') = \emptyset$), for instance $\pm R^+$, $\pm R_I^+$ and $\pm R_L^+$, the multiplication induces, for any ordering of R' , an isomorphism of schemes over F

$$(2.2) \quad \prod_{\alpha \in R'} U_{\alpha} \xrightarrow{\simeq} U(R').$$

2.2. The parabolic big cell. We have the Bruhat decomposition of G ([3] Part II.1.9)

$$G = \bigcup_{w \in W} B^- w B^+ = \bigcup_{w \in W} U_G^- w T U_G^+.$$

Let C denote the parabolic big cell

$$C = \bigcup_{w \in W_I} U_G^- w T U_G^+ = U^- \left(\bigcup_{w \in W_I} U_L^- w T U_L^+ \right) U^+ = U^- L U^+ = U^- P^+ = P^- U^+.$$

Then G is the disjoint union of C and $U_G^- w T U_G^+$ for all $w \notin W_I$.

Let $r = |R_I^+| = \dim U^+$. We consider the adjoint representation of G on $\bigwedge^r \mathfrak{g}$ over F . From the decomposition (2.1) of \mathfrak{g} we obtain a direct sum decomposition of $\bigwedge^r \mathfrak{g}$. Choosing X_α a nonzero element in \mathfrak{g}_α for each $\alpha \in R$ and a basis of \mathfrak{g}_0 we obtain a basis of $\bigwedge^r \mathfrak{g}$ with respect to this decomposition and containing $Y = \bigwedge_{\alpha \in R_I^+} X_\alpha$. For $g \in G$ we define $f(g)$ to be the coefficient of Y in the expansion of $\text{Ad}(g)Y$ in the chosen basis of $\bigwedge^r \mathfrak{g}$. Then f is a regular function on G over F .

There is a partial order on $\mathbb{Z}S$: $\gamma < \delta$ iff $\delta - \gamma$ is a sum of positive roots. If one considers the group action of the symmetric group S_r on $(\mathbb{Z}S)^r$ via coordinate permutation, the set of the unordered r -tuples $[\gamma_1, \dots, \gamma_r]$ of elements in $\mathbb{Z}S$ may be viewed as the set of S_r -orbits in $(\mathbb{Z}S)^r$. We define $[\gamma_1, \dots, \gamma_r] < [\delta_1, \dots, \delta_r]$ iff there exists $s \in S_r$ such that $\gamma_j \leq \delta_{s(j)}$ for all $1 \leq j \leq r$ and $\gamma_j < \delta_{s(j)}$ for at least one j .

We adopt the convention that $\mathfrak{g}_\gamma = 0$ if $\gamma \in \mathbb{Z}S$ is nonzero and not a root. Then $[\mathfrak{g}_\beta, \mathfrak{g}_\alpha] \subset \mathfrak{g}_{\alpha+\beta}$, and therefore

$$\text{Ad}(u_\beta)X_\alpha \in X_\alpha + \sum_{i \geq 1} \mathfrak{g}_{\alpha+i\beta}, \quad u_\beta \in U_\beta.$$

If $\alpha, \beta \in R_I^+$, then it is clear that either $\alpha + i\beta \in R_I^+$ or $\mathfrak{g}_{\alpha+i\beta} = 0$. The same statement holds for $\alpha \in R_I^+$ and $\beta \in R_I$, since the β -string through α lies in R_I^+ . Therefore U_β fixes Y for any $\beta \in R_I^+ \cup R_I = R^+ \cup (-R_I^+)$. (2.2) implies that

$$(2.3) \quad Y \text{ is invariant under } U_G^+ \text{ and } U_L^-.$$

If we let β be negative roots, then (2.2) also implies that for $v^- \in U_G^-$,

$$(2.4) \quad \text{Ad}(v^-)X_\alpha \in X_\alpha + \sum_{\gamma < \alpha} \mathfrak{g}_\gamma.$$

For $t \in T$,

$$(2.5) \quad \text{Ad}(t)Y = \prod_{\alpha \in R_I^+} \alpha(t)Y.$$

For $w \in W$ there exists a constant $c_{w,\alpha} \in F^\times$ satisfying

$$(2.6) \quad \text{Ad}(w)X_\alpha = c_{w,\alpha}X_{w\alpha}.$$

In view of (2.2), we see that w preserves R_I^+ iff w normalizes U^+ . Since $N_G(U^+) = P^+$ and $w \in P^+$ iff $w \in W_I$,

$$(2.7) \quad w \text{ preserves } R_I^+ \text{ iff } w \in W_I.$$

Since $P^+ = \bigcup_{w \in W_I} U_L^- w T U_G^+$, if we write $p^+ = u^- w t v^+$ ($v^+ \in U_G^+$, $t \in T$, $w \in W_I$, $u^- \in U_L^-$), then it follows from (2.3), (2.5), (2.6) and (2.7) that

$$\text{Ad}(p^+)Y = \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha}(t) \cdot Y,$$

where $\text{sign}(w)$ denotes the sign of the permutation w on R_I^+ . Moreover, it follows from (2.4) that

$$\text{Ad}(v^- w t v^+)Y \in \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha}(t) \cdot Y + \sum_{[\gamma_j] < [\alpha]_{\alpha \in R_I^+}} \bigwedge_{j=1}^r \mathfrak{g}_{\gamma_j}.$$

So

$$f(v^- w t v^+) = \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha}(t).$$

Similarly, for $w \notin W_I$,

$$\text{Ad}(v^- w t v^+)Y \in \sum_{[\gamma_j] \leq [w\alpha]_{\alpha \in R_I^+}} \bigwedge_{j=1}^r \mathfrak{g}_{\gamma_j}.$$

It follows from the proof of (2.3) that $\alpha + \beta \in R_I^+$ or $\mathfrak{g}_{\alpha+\beta} = 0$ if $\alpha \in R_I^+$ and $\beta \in R^+$, so $\alpha \in R_I^+$ and $\delta \geq \alpha$ imply $\delta \in R_I^+$ or $\mathfrak{g}_\delta = 0$. Therefore if $\delta \in \{0\} \cup R - R_I^+$ and $\gamma \leq \delta$ then $\gamma \notin R_I^+$, and hence (2.7) implies that Y does not appear in the expression of $\text{Ad}(v^- w t v^+)Y$, so $f(v^- w t v^+) = 0$.

We conclude with the following lemma.

LEMMA 2.1. *Let the notations be as above, then*

(1) *For $p^+ \in P^+$,*

$$\text{Ad}(p^+)Y = f(p^+)Y,$$

and hence f is an F -rational character on P^+ .

(2) *For $w \in W_I$,*

$$f(v^- w t v^+) = \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha}(t), \quad v^\pm \in U_G^\pm, \quad t \in T.$$

(3) *f vanishes on $U_G^- w T U_G^+$ for $w \notin W_I$.*

In particular,

(4) *C is an open F -subscheme of G , and $F[C] = F[G]_f$.*

(5) *f is right invariant under U_G^+ and left invariant under U_G^- .*

EXAMPLE 2.2. [cf. [3] Part II 1.9 and [13] §5 Theorem 7] *If $I = \emptyset$, then $L = T$ and $U^\pm = U_G^\pm$. Lemma 2.1 implies that $f(u^- t u^+) = \prod_{\alpha \in R^+} \alpha(t)$ and $f(u^- w t u^+) = 0$ for all nontrivial w , $u^\pm \in U_G^\pm$ and $t \in T$.*

EXAMPLE 2.3. Let $G = \mathrm{SL}(n+1, F)$. Write $g = \begin{pmatrix} A & B \\ C & d \end{pmatrix}$ with $A \in \mathrm{M}(n, F), B \in \mathrm{M}(n, 1; F), C \in \mathrm{M}(1, n; F)$ and $d \in F$. Let

$$\begin{aligned} U^+ &= \left\{ \begin{pmatrix} I_n & 0 \\ C & 1 \end{pmatrix} \in G \right\}, \\ L &= \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in G \right\}. \end{aligned}$$

Some calculations show that

$$f(g) = d^{n+1}.$$

EXAMPLE 2.4. More generally, we consider $G = \mathrm{GL}(n, F)$. Let (n_1, \dots, n_s) be a partition of n . Write $g = (g_{ij})_{1 \leq i, j \leq s}$ with $g_{ij} \in \mathrm{M}(n_i, n_j; F)$. Let U^+ be the subgroup consisting of the matrices $u = (u_{ij})$ such that $u_{ij} = 0$ for $j < i$ and $u_{ii} = I_{n_i}$, and L the subgroup consisting of the matrices $l = (l_{ij})$ such that $l_{ij} = 0$ for $i \neq j$. $L \cong \prod_{1 \leq i \leq s} \mathrm{GL}(n_i, F)$.

For $l \in L$,

$$\begin{aligned} f(l) &= \prod_{1 \leq j < i \leq s} \det(l_{ii})^{n_j} \det(l_{jj})^{-n_i} \\ &= \prod_{1 \leq i \leq s} \det(l_{ii})^{\sum_{j < i} n_j - \sum_{i < k} n_k}. \end{aligned}$$

The computation of the explicit formula for f involves the process of block lower (or upper) triangularization, and it turns out to be complicated if $s \geq 3$. For $s = 2$, we have

$$f(g) = \det(g_{22})^{n_1+n_2} \det(g)^{-n_2}.$$

The situation is the same if $G = \mathrm{SL}(n, F)$. For instance, if $s = 2$, then

$$f(g) = \det(g_{22})^{n_1+n_2}.$$

EXAMPLE 2.5. We consider

$$G = \mathrm{Sp}(2n, F) = \left\{ g \in \mathrm{GL}(2n, F) : {}^t g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in \mathrm{M}(n, F)$. Let

$$\begin{aligned} U^+ &= \left\{ \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} : C \in \mathrm{Sym}(n, F) \right\}, \\ L &= \left\{ \begin{pmatrix} {}^t D^{-1} & 0 \\ 0 & D \end{pmatrix} : D \in \mathrm{GL}(n, F) \right\}, \end{aligned}$$

where $\mathrm{Sym}(n, F)$ is the group of symmetric matrices of order n over F . Some calculations show that

$$f(g) = \det(D)^{n+1}.$$

3. $\text{Ind}_P^G \sigma$ and the principal series $(C_\sigma^{\text{an}}(\mathfrak{H}, V), T_\sigma)$

Induced representations, especially the parabolic inductions, are of extreme importance in Lie theory. For p -adic Lie groups, they were studied by Féaux de Lacroix in his work ([2]) on the locally analytic representations.

We first recall the notion of locally analytic representations over K of an F -Lie group.

DEFINITION 3.1 (cf. [2] and [11] §3). *A locally analytic representation (σ, V) of an F -Lie group G on a barreled locally convex Hausdorff K -vector space V is a continuous representation such that the orbit maps are V -valued locally analytic functions on G . More precisely, for any $v \in V$ there exists a BH-space W of V (that is, a Banach space W together with a continuous injection $W \hookrightarrow V$) such that $g \mapsto \sigma(g)v$ expands (in a neighborhood of the unit element) to a power series with W -coefficients.*

Let (σ, V) be a locally analytic representation of the Levi subgroup L . σ extends to a representation of P^- defined by $\sigma(ul) = \sigma(l)$ ($l \in L, u \in U^-$).

DEFINITION 3.2. *Let $\text{Ind}_P^G \sigma$ be the space of V -valued locally analytic functions ϕ on G satisfying*

$$\phi(p^-g) = \sigma(p^-)\phi(g), \quad \text{for all } g \in G, p^- \in P^-.$$

On $\text{Ind}_P^G \sigma$ we have a G -action defined by right translation.

Since the quotient space $P^- \backslash G$ is compact, we obtain from [2] 4.1.5 the following proposition.

PROPOSITION 3.3. *$\text{Ind}_P^G \sigma$ is a locally analytic representation of G .*

Next, we introduce the principal series representation, serving as another description of $\text{Ind}_P^G \sigma$.

Let \mathfrak{H} and $\overline{\mathfrak{H}}$ denote the G -homogeneous spaces $U^- \backslash G$ and $P^- \backslash G$ respectively, and denote $\hat{g} := \text{pr}_{\mathfrak{H}}^G(g)$. Because $P^- \cong U^- \rtimes L$, there is a left L -action on \mathfrak{H} , and $\overline{\mathfrak{H}} = L \backslash \mathfrak{H}$.

DEFINITION 3.4. *Let $C_\sigma^{\text{an}}(\mathfrak{H}, V)$ be the space of V -valued locally analytic functions φ on \mathfrak{H} satisfying*

$$\varphi(l\hat{g}) = \sigma(l)\varphi(\hat{g}), \quad \text{for all } \hat{g} \in \mathfrak{H} \text{ and } l \in L.$$

The principal series representation $(C_\sigma^{\text{an}}(\mathfrak{H}, V), T_\sigma)$ of G is defined via

$$(T_\sigma(g)\varphi)(\widehat{g'}) := \varphi(\widehat{g'} \cdot g).$$

LEMMA 3.5.

- (1) *The representation $\text{Ind}_P^G \sigma$ is (naturally) isomorphic to $(C_\sigma^{\text{an}}(\mathfrak{H}, V), T_\sigma)$.*
- (2) *$\text{Ind}_P^G \sigma$ is isomorphic to $C^{\text{an}}(\overline{\mathfrak{H}}, V)$.*
- (3) *Let ι be a locally analytic section of $\text{pr}_{\mathfrak{H}}^G$ then ι induces an isomorphism*

$$\begin{aligned} \iota^\circ : C_\sigma^{\text{an}}(\mathfrak{H}, V) &\rightarrow C^{\text{an}}(\overline{\mathfrak{H}}, V) \\ \varphi &\mapsto \varphi \circ \iota. \end{aligned}$$

PROOF. (1) From a locally analytic section $\bar{\iota}$ of $\text{pr}_{\mathfrak{S}}^G$ we obtain an isomorphism ([2] 4.3.1)

$$\bar{\iota}^\circ : \text{Ind}_{\mathbf{U}^-}^G \mathbf{1} \simeq C^{\text{an}}(\mathfrak{S}, V), \quad \phi \mapsto \phi \circ \bar{\iota}.$$

By restriction to the subspaces, $\bar{\iota}^\circ$ induces an isomorphism, independent of $\bar{\iota}$, between $\text{Ind}_{\mathbf{P}^-}^G \sigma$ and $C_\sigma^{\text{an}}(\mathfrak{S}, V)$. G -equivariance is evident.

(2) A locally analytic section $\tilde{\iota}$ of $\text{pr}_{\mathfrak{S}}^G$ induces an isomorphism $\tilde{\iota}^\circ$ from $\text{Ind}_{\mathbf{P}^-}^G \sigma$ onto $C^{\text{an}}(\overline{\mathfrak{S}}, V)$ (ibid.).

(3) Choose $\bar{\iota}$ and $\tilde{\iota}$ compatible with ι , that is, $\tilde{\iota} = \bar{\iota} \circ \iota$, then the assertion follows from (1) and (2). Q.E.D.

Compactness of $\overline{\mathfrak{S}}$ implies that $C^{\text{an}}(\overline{\mathfrak{S}}, V)$ is of compact type ([11] Lemma 2.1). By [11] Proposition 1.2, Theorem 1.3 and [9] Proposition 16.10, we have the following corollary.

COROLLARY 3.6. *Suppose that K is spherically complete. Let B be a closed subspace of $C_\sigma^{\text{an}}(\mathfrak{S}, V)$, then both B and $C_\sigma^{\text{an}}(\mathfrak{S}, V)/B$ are of compact type. In particular, they are reflexive, bornological, and complete. Moreover, their strong duals B_b^* and $(C_\sigma^{\text{an}}(\mathfrak{S}, V)/B)_b^*$ are nuclear Fréchet spaces.*

4. Rigid analytic symmetric space Ω

The rigid analytic symmetric space Ω associated to G (with respect to a parabolic \mathbf{P}^+) was constructed by van der Put and Voskuil in [14]. Some examples are the p -adic upper half-plane, Drinfel'd's space and the p -adic Siegel upper half-space, which are associated to $\text{SL}(2, F)$, $\text{SL}(n+1, F)$ and $\text{Sp}(2n, F)$ respectively (cf. [4], [10] and [7]).

4.1. Definition of the symmetric space Ω . Let \mathbf{G} , \mathbf{P}^\pm , \mathbf{U}^\pm , \mathbf{L} and \mathbf{C} denote the F -rigid analytifications of G , P^\pm , U^\pm , L and C respectively. f defines a rigid analytic function on \mathbf{G} .

Since f is left invariant under \mathbf{U}^- (Lemma 2.1 (5)), we may define

$$f(\hat{g}, \mathbf{u}) := f(g \cdot \mathbf{u})$$

for $\hat{g} \in \mathfrak{S}$ and $\mathbf{u} \in \mathbf{U}^-$.

DEFINITION 4.1. *Let*

$$\begin{aligned} \Omega &:= \{\mathbf{u} \in \mathbf{U}^- : g \cdot \mathbf{u} \in \mathbf{C}, \text{ for all } g \in \mathbf{G}\} \\ &= \{\mathbf{u} \in \mathbf{U}^- : f(\hat{g}, \mathbf{u}) \neq 0, \text{ for all } \hat{g} \in \mathfrak{S}\}. \end{aligned}$$

We call Ω the symmetric space associated to G with respect to \mathbf{P}^+ .

EXAMPLE 4.2. *In the situation of Example 2.3, $\mathbf{U}^- \cong \mathbf{A}_{/F}^n$ and $(z_1, \dots, z_n) \in \Omega$ is given by the inequalities*

$$c_1 z_1 + \dots + c_n z_n + d \neq 0 \quad \text{for all nonzero } (c_1, \dots, c_n, d) \in F^{n+1}.$$

Therefore Ω is Drinfel'd's space.

EXAMPLE 4.3. In the situation of Example 2.5, $\mathbf{U}^- \cong \mathbf{Sym}(n)$ and $Z \in \Omega$ is given by the inequalities

$$\det(CZ + D) \neq 0 \quad \text{for all } C^t D = D^t C, \text{ rank}(C \ D) = n.$$

Therefore Ω is the p -adic Siegel upper half-space.

We may also interpret Ω to be the complement of all the G -translations of $(\mathbf{G}-\mathbf{C})/\mathbf{P}^+ = \mathbf{G}/\mathbf{P}^+ - \mathbf{U}^-$ in \mathbf{G}/\mathbf{P}^+ . Therefore we have a left G -action on Ω (induced from the left G -action on \mathbf{G}/\mathbf{P}^+). We denote $g * \mathbf{u}$ the action of $g \in G$ on $\mathbf{u} \in \Omega$. We have $g * \mathbf{u} = \text{pr}_{\mathbf{U}^-}^{\mathbf{C}}(g \cdot \mathbf{u})$.

4.2. Automorphy factor. We define the *automorphy factor*

$$\begin{aligned} j : G \times \Omega &\rightarrow \mathbf{P}^+ \\ (g, \mathbf{u}) &\mapsto (g * \mathbf{u})^{-1} \cdot g \cdot \mathbf{u}. \end{aligned}$$

Then $j(g, \mathbf{u}) = \text{pr}_{\mathbf{P}^+}^{\mathbf{C}}(g \cdot \mathbf{u})$, and straightforward computations show

$$(4.1) \quad j(g_1 g_2, \mathbf{u}) = j(g_1, g_2 * \mathbf{u}) j(g_2, \mathbf{u}).$$

For any $u \in \mathbf{U}^-$, $j(u, \mathbf{u}) = 1_{\mathbf{G}}$, and hence (4.1) implies $j(u \cdot g, \mathbf{u}) = j(g, \mathbf{u})$, so we may define $j(\hat{g}, \mathbf{u}) := j(g, \mathbf{u})$.

For $l \in L$, since $l * \mathbf{u} = l \cdot \mathbf{u} \cdot l^{-1}$, we have $j(l, \mathbf{u}) = l$, and by (4.1)

$$(4.2) \quad j(l \cdot \hat{g}, \mathbf{u}) = l \cdot j(\hat{g}, \mathbf{u}).$$

Since f is left invariant under \mathbf{U}^- (Lemma 2.1(5)), it follows that

$$(4.3) \quad f(j(\hat{g}, \mathbf{u})) = f(\hat{g}, \mathbf{u}).$$

From Lemma 2.1 (1), (4.1) and (4.3), we see that

$$(4.4) \quad f(\widehat{g_1 g_2}, \mathbf{u}) = f(\widehat{g_1}, g_2 * \mathbf{u}) f(\widehat{g_2}, \mathbf{u}).$$

Moreover, Lemma 2.1 (1), (4.2) and (4.3) imply

$$(4.5) \quad f(l \cdot \hat{g}, \mathbf{u}) = f(l) f(\hat{g}, \mathbf{u}).$$

4.3. The F -rigid analytic structure on Ω . van der Put and Voskuil defined an affinoid covering of Ω using Bruhat-Tits Buildings ([14]). In this paper we choose another approach following the construction of affinoid covering of Drinfel'd's space in [10] and that of p -adic Siegel upper half-space in [7]. We endow Ω with a structure of F -rigid analytic variety and show that it is an admissible open subset of \mathbf{U}^- and, in particular, an open rigid analytic subspace of \mathbf{U}^- (and therefore \mathbf{G}/\mathbf{P}^+).

We realize G as a Zariski closed subgroup of $\text{GL}(n, F)$ such that T consists of diagonal matrices and B^+ consists of lower triangular matrices. Then $f(g)$ extends to an F -regular function on $\text{GL}(n, F)$ with respect to the coordinates $g_{i,j}$ ($1 \leq i, j \leq n$), and $\det(g)^r f(g)$ is a homogeneous F -polynomial in $g_{i,j}$. We denote N the degree of $\det(g)^r f(g)$ and let M be an integer such that all the coefficients have absolute values bounded by $|\varpi|^{NM}$.

LEMMA 4.4. Ω is nonempty.

PROOF. It suffices to prove that G -translations of $\mathbf{G} - \mathbf{C}$ do not cover \mathbf{G} . For $g \in G$, $g \cdot (\mathbf{G} - \mathbf{C})$ is the locus of $f(g^{-1} \cdot \mathbf{g}) = 0$. With the embedding of \mathbf{G} into $\mathbf{GL}(n, F)$ we view $f(g^{-1} \cdot \mathbf{g})$ as a rational function in \mathbf{g}_{ij} with $F[G]$ -coefficients, and, for a given $g \in G$, $f(g^{-1} \cdot \mathbf{g})$ is a nonzero F -rational function in \mathbf{g}_{ij} . It is not hard to see that there are choices of $\mathbf{g}_{ij} \in F^{\text{alg}}$ with appropriate absolute values so that the non-vanishing monomials in $f(g^{-1} \cdot \mathbf{g})$ are of distinct absolute values in $|(F^{\text{alg}})^{\times}| = |\varpi|^{\mathbb{Q}}$ modulo $|F^{\times}| = |\varpi|^{\mathbb{Z}}$. Therefore there exists $\mathbf{g} \in \mathbf{G}$ such that $f(g^{-1} \cdot \mathbf{g})$ is nonzero for all $g \in G$, and consequently \mathbf{g} lies in the complement of all the $g \cdot (\mathbf{G} - \mathbf{C})$. Q.E.D.

Let G_0 and L_0 denote the intersections of G and L with $\mathbf{GL}(n, \mathfrak{o})$ respectively, and denote $\mathfrak{H}_0 = \text{pr}_{\mathfrak{H}}^G(G_0)$.

We recall Iwasawa's decomposition

$$G = B^- G_0.$$

Then $G = P^- \cdot G_0$ and $\mathfrak{H} = L \cdot \mathfrak{H}_0$, so (4.5) and Lemma 2.1 imply

$$\Omega = \{\mathbf{u} \in \mathbf{U}^- : f(\hat{g}, \mathbf{u}) \neq 0, \text{ for any } \hat{g} \in \mathfrak{H}_0\}.$$

For $\mathbf{u} \in \mathbf{U}^-$ an upper triangular matrix with diagonal entries 1, let

$$|\mathbf{u}| := \max_{1 \leq i \leq j \leq n} |\mathbf{u}_{ij}| = \max_{1 \leq i < j \leq n} \{1, |\mathbf{u}_{ij}|\}.$$

For any nonnegative integer m and $\hat{g} \in \mathfrak{H}_0$, we define

$$\mathbf{B}(m; \hat{g}) := \{\mathbf{u} \in \mathbf{U}^- : |f(\hat{g}, \mathbf{u})| < |\mathbf{u}|^N |\varpi|^{N(M+m)}\}.$$

LEMMA 4.5. *If m is a nonnegative integer and $g_1, g_2 \in G_0$ such that $g_1 \equiv l \cdot g_2 \pmod{\varpi^{Nm+1}}$ for some $l \in L_0$, then*

$$\mathbf{B}(m; \widehat{g_1}) = \mathbf{B}(m; \widehat{g_2}).$$

PROOF. Since $f|_{L_0}$ is a continuous F -character (Lemma 2.1 (1)) and L_0 is compact, the image of L_0 under f is contained in \mathfrak{o}^{\times} . Therefore (4.5) implies $|f(\widehat{lg_2}, \mathbf{u})| = |f(\widehat{g_2}, \mathbf{u})|$, and hence $\mathbf{B}(m; \widehat{g_2}) = \mathbf{B}(m; \widehat{lg_2})$. So we may assume $g_1 \equiv g_2 \pmod{\varpi^{Nm+1}}$.

We choose $\lambda \in (F^{\text{alg}})^{\times}$ such that $|\lambda| = |\mathbf{u}|$. Since $|\lambda^{-1} \mathbf{u}_{ij}| \leq 1$,

$$g_1 \cdot \lambda^{-1} \mathbf{u} \equiv g_2 \cdot \lambda^{-1} \mathbf{u} \pmod{\varpi^{Nm+1}},$$

and the matrices on both sides have entries with absolute values ≤ 1 . Applying $\det' \cdot f$, we obtain

$$\lambda^{-N} \det(g_1)^r f(g_1 \cdot \mathbf{u}) \equiv \lambda^{-N} \det(g_2)^r f(g_2 \cdot \mathbf{u}) \pmod{\varpi^{NM+Nm+1}},$$

and consequently

$$|\mathbf{u}|^{-N} |f(\widehat{g_1}, \mathbf{u})| < |\varpi|^{N(M+m)} \Leftrightarrow |\mathbf{u}|^{-N} |f(\widehat{g_2}, \mathbf{u})| < |\varpi|^{N(M+m)}.$$

Therefore $\mathbf{B}(m; \widehat{g_1}) = \mathbf{B}(m; \widehat{g_2})$.

Q.E.D.

Let

$$\begin{aligned}\Omega(m; \hat{g}) &:= \mathbf{U}^- - \mathbf{B}(m; \hat{g}) \\ &= \left\{ \mathbf{u} \in \mathbf{U}^- : |f(\hat{g}, \mathbf{u})| \geq |\mathbf{u}_{ij}|^N |\varpi|^{N(M+m)}, 1 \leq i \leq j \leq n \right\}, \\ \Omega(m) &:= \bigcap_{\hat{g} \in \mathfrak{H}_0} \Omega(m; \hat{g}).\end{aligned}$$

For a given $\mathbf{u} \in \Omega$, $|f(\hat{g}, \mathbf{u})|$ has a positive lower bound on \mathfrak{H}_0 . Therefore

$$\Omega = \bigcup_{m=0}^{\infty} \Omega(m).$$

Let $\mathfrak{H}^{(m)}$ be any finite subset of \mathfrak{H}_0 including a set of representatives in \mathfrak{H}_0 for $\text{pr}_{\mathfrak{S}}^G(\mathbf{L}_0 \backslash \mathbf{G}_0 / \mathbf{G}_0(Nm+1))$, where $\mathbf{G}_0(Nm+1)$ denotes the congruence subgroup $(I_n + \varpi^{Nm+1} \mathbf{M}(n, \mathfrak{o})) \cap \mathbf{G}$. Then Lemma 4.5 implies that

$$\Omega(m) = \bigcap_{\hat{g} \in \mathfrak{H}^{(m)}} \Omega(m; \hat{g}).$$

Moreover, we may assume that $\mathfrak{H}^{(m)}$ contains \hat{I}_n .

$$\Omega(m; \hat{I}_n) = \left\{ \mathbf{u} \in \mathbf{U}^- : |\varpi^{M+m} \mathbf{u}_{ij}| \leq 1 \right\}$$

is an admissible open affinoid subset of \mathbf{U}^- . $\Omega(m)$ is the intersection of finitely many rational sub-domains of $\Omega(m; \hat{I}_n)$:

$$\left\{ \mathbf{u} \in \Omega(m; \hat{I}_n) : \left| \frac{\varpi^{N(M+m)} \mathbf{u}_{ij}^N}{f(\hat{g}, \mathbf{u})} \right| \leq 1, 1 \leq i \leq j \leq n \right\},$$

with \hat{g} ranging on $\mathfrak{H}^{(m)} - \{\hat{I}_n\}$. Therefore $\Omega(m)$ is an affinoid variety.

We conclude that $\{\Omega(m)\}_{m=0}^{\infty}$ constitutes an admissible affinoid covering of Ω so that Ω admits a rigid analytic variety structure (see [1] 9.3). According to [1] 9.1.2 Lemma 3 (compare [1] 9.1.4 Proposition 2), the following proposition implies that Ω is an admissible open subset of \mathbf{U}^- .

PROPOSITION 4.6. *Any morphism from an affinoid variety to \mathbf{U}^- with image in Ω factors through some $\Omega(m)$.*

PROOF. The argument is similar to the third proof of [10] §1 Proposition 1.

Let \mathbf{X} be an affinoid variety, $\Phi : \mathbf{X} \rightarrow \mathbf{U}^-$ a morphism from \mathbf{X} to \mathbf{U}^- with image in Ω . For any $\hat{g} \in \mathfrak{H}_0$,

$$\mathbf{x} \mapsto \frac{\Phi(\mathbf{x})_{ij}^N}{f(\hat{g}, \Phi(\mathbf{x}))}, \quad 1 \leq i \leq j \leq n,$$

are F -rigid analytic functions on \mathbf{X} . By the maximum modulus principle ([1] §6.2 Proposition 4 (i)), there exists a positive integer $m_{\hat{g}}$ such that

$$\max_{1 \leq i \leq j \leq n} \max_{\mathbf{x} \in \mathbf{X}} \left| \frac{\Phi(\mathbf{x})_{ij}^N}{f(\hat{g}, \Phi(\mathbf{x}))} \right| \leq |\varpi|^{-N(M+m_{\hat{g}})}.$$

In other words, $\Phi(\mathbf{X}) \subset \Omega(m_{\hat{g}}; \hat{g})$. In view of Lemma 4.5, $m_{\hat{g}}$ can be chosen locally constant. Therefore the compactness of \mathfrak{S}_0 implies that there exists a positive integer m such that $\Phi(\mathbf{X}) \subset \Omega(m)$. Q.E.D.

Finally, we prove that the morphisms of g -translations from $\Omega(m)$ into Ω indeed factor through the same $\Omega(m')$ for all $g \in G_0$.

LEMMA 4.7. *For any nonnegative integer m , there exists a nonnegative integer m' such that for all $g \in G_0$,*

$$g * \Omega(m) \subset \Omega(m').$$

PROOF. Let $\mathbf{u} \in \Omega(m)$. Then

$$(4.6) \quad 1 \leq |\mathbf{u}| \leq |\varpi|^{-M-m},$$

and

$$(4.7) \quad \frac{|\mathbf{u}|^N}{|f(\hat{g}, \mathbf{u})|} \leq |\varpi|^{-N(M+m)} \quad \text{for any } g \in G_0.$$

$g * \mathbf{u} = \text{pr}_{\mathbf{U}}^{\mathbf{C}}(g \cdot \mathbf{u})$, and since $\text{pr}_{\mathbf{U}}^{\mathbf{C}}$ is F -regular on \mathbf{C} , Lemma 2.1 (4) implies that there exist positive integers s and t such that all the entries of

$$\det(g \cdot \mathbf{u})^t f(g \cdot \mathbf{u})^s \cdot g * \mathbf{u} = \det(g)^t f(g \cdot \mathbf{u})^s \cdot g * \mathbf{u}$$

are F -polynomials with variables the entries of $g \cdot \mathbf{u}$. Let D be the highest degree and L an integer such that the absolute values of all the coefficients are bounded by $|\varpi|^L$. Since $g \in G_0$, the entries of $g \cdot \mathbf{u}$ have absolute values $\leq |\mathbf{u}|$ and $|\det(g)| = 1$, then

$$(4.8) \quad |g * \mathbf{u}| \leq \frac{|\varpi|^L |\mathbf{u}|^D}{|f(\hat{g}, \mathbf{u})|^s}.$$

It follows from (4.4), (4.8), (4.6) and (4.7) that for any $g_1 \in G_0$,

$$\begin{aligned} \frac{|g * \mathbf{u}|^N}{|f(\hat{g}_1, g * \mathbf{u})|} &\leq \frac{|\varpi|^{NL} |\mathbf{u}|^{ND}}{|f(\hat{g}_1, \mathbf{u})| |f(\hat{g}, \mathbf{u})|^{Ns-1}} \\ &= |\varpi|^{NL} |\mathbf{u}|^{ND-N^2s} \frac{|\mathbf{u}|^N}{|f(\hat{g}_1, \mathbf{u})|} \frac{|\mathbf{u}|^{N^2s-N}}{|f(\hat{g}, \mathbf{u})|^{Ns-1}}, \\ &\leq |\varpi|^{N(L-\max\{D, Ns\}(M+m))}. \end{aligned}$$

Therefore $g * \Omega(m) \subset \Omega(m')$ for any $m' \geq -M - L + \max\{D, Ns\}(M + m)$. Q.E.D.

4.4. Rigid analytic functions on Ω . Let $\mathcal{O}(\Omega(m))$ denote the space of F -rigid analytic functions on $\Omega(m)$. Then $\mathcal{O}(\Omega(m))$ is an F -affinoid algebra with the supremum norm.

Let $\mathcal{O}(\Omega)$ be the F -algebra of F -rigid analytic functions on Ω , that is, the projective limit of $\mathcal{O}(\Omega(m))$,

$$\mathcal{O}(\Omega) := \varprojlim_m \mathcal{O}(\Omega(m)).$$

$\mathcal{O}(\Omega)$ is endowed with the projective limit topology.

From the construction of $\Omega(m)$ we see that the F -affinoid algebra $\mathcal{O}(\Omega(m))$ is equal to

$$(4.9) \quad F \left\langle \varpi^{M+m} \mathbf{u}_{ij}, \frac{\varpi^{N(M+m)} \mathbf{u}_{ij}^N}{f(\hat{g}, \mathbf{u})} : 1 \leq i \leq j \leq n, \hat{g} \in \mathfrak{S}^{(m)} - \{\hat{f}_n\} \right\rangle.$$

Therefore $\psi \in \mathcal{O}(\Omega(m))$ has an expansion in the following form that converges with respect to the supremum norm $\|\cdot\|_{\mathcal{O}(\Omega(m))}$:

$$(4.10) \quad \psi(\mathbf{u}) = \sum_{(\ell_{\hat{g}}) \in (\mathbb{N}_0)^{\mathfrak{S}^{(m)}}} P_{(\ell_{\hat{g}})}(\mathbf{u}) \prod_{\hat{g} \in \mathfrak{S}^{(m)}} f(\hat{g}, \mathbf{u})^{-\ell_{\hat{g}}},$$

where $P_{(\ell_{\hat{g}})}(\mathbf{u})$ are polynomials in the coordinates of \mathbf{u} with coefficients in F .

In view of (4.5), the assumption that $\mathfrak{S}^{(m)}$ is contained in \mathfrak{S}_0 is quite artificial, and it is more convenient and natural to choose $\mathfrak{S}^{(m)}$ to be an arbitrary finite subset of \mathfrak{S} whenever we consider the expansion of $\psi \in \mathcal{O}(\Omega(m))$.

$\psi \in \mathcal{O}(\Omega)$ may be considered as an F^{alg} -valued function on Ω such that, restricting on each $\Omega(m)$, ψ has an expansion (4.10) that converges with respect to $\|\cdot\|_{\mathcal{O}(\Omega(m))}$. In particular, $f(\hat{g}, \mathbf{u})^{-1} \in \mathcal{O}(\Omega)$ for any $\hat{g} \in \mathfrak{S}$.

Since all the generators of $\mathcal{O}(\Omega(m))$ in (4.9) are F -rigid analytic functions on $\Omega(m')$ for any $m' \geq m$ and therefore on Ω , we obtain the following proposition.

PROPOSITION 4.8.

- (1) Ω is a Stein space, that is, the image of $\mathcal{O}(\Omega(m+1))$ under the transition homomorphism in $\mathcal{O}(\Omega(m))$ is dense for any nonnegative integer m .
- (2) The image of $\mathcal{O}(\Omega)$ under the transition homomorphism in $\mathcal{O}(\Omega(m))$ is dense.

Let $\mathcal{O}_K(\Omega(m))$ and $\mathcal{O}_K(\Omega)$ denote $\mathcal{O}(\Omega(m)) \hat{\otimes}_F K$ and $\mathcal{O}(\Omega) \hat{\otimes}_F K$ respectively. If we let $\Omega_K(m)$ and Ω_K denote the extensions of the ground field K/F of $\Omega(m)$ and Ω respectively (see [1] §9.3.6), then $\mathcal{O}_K(\Omega(m))$ and $\mathcal{O}_K(\Omega)$ are the spaces of K -rigid analytic functions on $\Omega_K(m)$ and Ω_K respectively.

PROPOSITION 4.9. *Let K be spherically complete. $\mathcal{O}_K(\Omega)$ is a nuclear K -Fréchet space.*

PROOF. By [9] Proposition 19.9, it suffices to prove that all the $\mathcal{O}_K(\Omega(m))$ constitute a compact projective system.

Consider the $\mathcal{O}_K(\Omega(m-1))$ -norms of the generators of $\mathcal{O}_K(\Omega(m))$ (see 4.9), then

$$\sup_{\mathbf{u} \in \Omega(m-1)} \max_{\hat{g} \in \mathfrak{S}^{(m)} - \{\hat{f}_n\}} \max_{1 \leq i \leq j \leq n} \left\{ \left| \varpi^{M+m} \mathbf{u}_{ij} \right|, \left| \frac{\varpi^{N(M+m)} \mathbf{u}_{ij}^N}{f(\hat{g}, \mathbf{u})} \right| \right\} \leq |\varpi|.$$

[12] Lemma 1.5 implies that the transition homomorphism from $\mathcal{O}_K(\Omega(m))$ to $\mathcal{O}_K(\Omega(m-1))$ is compact. Q.E.D.

[11] Theorem 1.3 and Proposition 1.2 imply the following corollary.

COROLLARY 4.10. *Suppose K is spherically complete. Let \mathcal{N} be a closed subspace of $\mathcal{O}_K(\Omega)$, then \mathcal{N} and $\mathcal{O}_K(\Omega)/\mathcal{N}$ are nuclear Fréchet spaces, and their strong duals \mathcal{N}_b^* and $(\mathcal{O}_K(\Omega)/\mathcal{N})_b^*$ are of compact type.*

5. Holomorphic discrete series $(\mathcal{O}_\sigma(\Omega), \pi_\sigma)$

Let Ω (resp. $\Omega(m)$) denote $\mathbf{\Omega}_K(K)$ (resp. $\mathbf{\Omega}_K(m)(K)$). Restricting to Ω (resp. $\Omega(m)$), we view K -rigid analytic functions in $\mathcal{O}_K(\Omega)$ (resp. $\mathcal{O}_K(\Omega(m))$) as K -valued functions on Ω (resp. $\Omega(m)$), and abbreviate $\mathcal{O}_K(\Omega(m))$ (resp. $\mathcal{O}_K(\Omega)$) to $\mathcal{O}(\Omega(m))$ (resp. $\mathcal{O}(\Omega)$).

Let (V, σ) be a d -dimensional K -rational representation of L . σ extends to a representation of P^+ .

Let $\mathcal{O}_\sigma(\Omega) := \mathcal{O}(\Omega) \otimes_K V$ and $\mathcal{O}_\sigma(\Omega(m)) := \mathcal{O}(\Omega(m)) \otimes_K V$.

For any $g \in G$ and $\psi \in \mathcal{O}_\sigma(\Omega)$, let $\pi_\sigma(g)\psi$ be the V -valued function on Ω as follows

$$(\pi_\sigma(g)\psi)(\mathbf{u}) := \sigma(j(g^{-1}, \mathbf{u}))^{-1} \psi(g^{-1} * \mathbf{u}).$$

LEMMA 5.1. $\pi_\sigma(g)\psi \in \mathcal{O}_\sigma(\Omega)$.

PROOF. Since σ is K -rational, each coordinate of $\sigma(j(g^{-1}, \mathbf{u}))^{-1}$ is a product of a K -polynomial in the coordinates of $j(g^{-1}, \mathbf{u})$ and a power of $\det(j(g^{-1}, \mathbf{u}))^{-1} = \det(g)$. Note that $j(g^{-1}, \mathbf{u}) = \text{pr}_{P^+}^C(g^{-1} \cdot \mathbf{u})$, and since $\text{pr}_{P^+}^C$ is F -regular on C , each coordinate of $j(g^{-1}, \mathbf{u})$ is a product of an F -polynomial in the coordinates of $g^{-1} \cdot \mathbf{u}$ and powers of $\det(g^{-1} \cdot \mathbf{u})^{-1} = \det(g)$ and $f(g^{-1} \cdot \mathbf{u})^{-1}$. Therefore each coordinate of $\sigma(j(g^{-1}, \mathbf{u}))^{-1}$ has a finite expansion of the form (4.10), and hence belongs to $\mathcal{O}(\Omega)$.

Similarly, the coordinates of $\psi(g^{-1} * \mathbf{u})$ also have expansions of the form (4.10). By Proposition 4.6, for any m , g^{-1} -translation maps $\Omega(m)$ into some $\Omega(m')$, and hence the norm of each coordinate of $\psi(g^{-1} * \mathbf{u})$ on $\Omega(m)$ is bounded by the norm of the corresponding coordinate of ψ on $\Omega(m')$. Therefore $\psi(g^{-1} * \mathbf{u}) \in \mathcal{O}_\sigma(\Omega)$.

We conclude that $\pi_\sigma(g)\psi \in \mathcal{O}_\sigma(\Omega)$.

Q.E.D.

It follows from the automorphy relation (4.1) that π_σ is an action of G on $\mathcal{O}_\sigma(\Omega)$.

DEFINITION 5.2. We call $(\mathcal{O}_\sigma(\Omega), \pi_\sigma)$ the holomorphic (rigid analytic) discrete series representation of G .

LEMMA 5.3. Let m and m' be as in Lemma 4.7. Then there exists a constant c depending on σ and m such that

$$\|\pi_\sigma(g)\psi\|_{\mathcal{O}_\sigma(\Omega(m))} \leq c \|\psi\|_{\mathcal{O}_\sigma(\Omega(m'))},$$

for all $g \in G_0$.

PROOF. The proof is similar to the arguments in Lemma 5.1, but instead of Proposition 4.6 we apply Lemma 4.7.

Using the expressions for the coordinates of $\sigma(j(g^{-1}, \mathbf{u}))^{-1}$ in the first paragraph of the proof of Lemma 5.1, we see that their $\mathcal{O}(\Omega(m))$ -norms are uniformly bounded on G_0 , so there is a constant $c > 0$ such that

$$\max_{g \in G_0} \max_{\mathbf{u} \in \Omega(m)} \|\sigma(j(g^{-1}, \mathbf{u}))^{-1}\|_{\text{End}(V)} \leq c.$$

Consequently,

$$\begin{aligned}
& \max_{g \in G_0} \|\pi_\sigma(g)\psi\|_{\mathcal{O}_\sigma(\Omega(m))} \\
&= \max_{g \in G_0} \max_{\mathbf{u} \in \Omega(m)} \|(\pi_\sigma(g)\psi)(\mathbf{u})\|_V \\
&\leq \max_{g \in G_0} \max_{\mathbf{u} \in \Omega(m)} \|\sigma(j(g^{-1}, \mathbf{u}))^{-1}\|_{\text{End}(V)} \cdot \max_{g \in G_0} \max_{\mathbf{u} \in \Omega(m)} \|\psi(g^{-1} * \mathbf{u})\|_V \\
&\leq c \max_{\mathbf{u} \in \Omega(m')} \|\psi(\mathbf{u})\|_V \\
&= c \|\psi\|_{\mathcal{O}_\sigma(\Omega(m'))}.
\end{aligned}$$

Q.E.D.

It follows from Lemma 5.3 that, for each m , the map

$$\begin{aligned}
G_0 \times \mathcal{O}_\sigma(\Omega) &\rightarrow \mathcal{O}_\sigma(\Omega(m)) \\
(g, \psi) &\mapsto (\pi_\sigma(g)\psi)|_{\Omega(m)}.
\end{aligned}$$

is continuous. Since $\mathcal{O}_\sigma(\Omega)$ is the projective limit of $\mathcal{O}_\sigma(\Omega(m))$, we obtain the following corollary.

COROLLARY 5.4. *$(\mathcal{O}_\sigma(\Omega), \pi_\sigma)$ is a continuous G -representation.*

Moreover, we shall prove that the dual representation of π_σ is locally analytic. For this, we recall that a coordinate chart at 1_G is obtained from the decomposition of the Bruhat big cell (see (2.2))

$$(5.1) \quad U^- U_L^- T U_L^+ U^+ \simeq \mathbb{A}_F^{|R|} \times \mathbb{G}_m(F)^{\dim \mathfrak{g}_0}.$$

LEMMA 5.5. *Let m and m' be as in Lemma 4.7. Let B be any parameterized (as in (5.1)) open neighborhood of 1_G contained in G_0 . For any $\psi \in \mathcal{O}_\sigma(\Omega(m'))$, the orbit map*

$$\begin{aligned}
B &\rightarrow \mathcal{O}_\sigma(\Omega(m)) \\
g &\mapsto (\pi_\sigma(g)\psi)|_{\Omega(m)}
\end{aligned}$$

is an $\mathcal{O}_\sigma(\Omega(m))$ -valued analytic function, namely, it can be expanded as a convergent power series with variables the coordinate parameters of B and coefficients in the Banach space $\mathcal{O}_\sigma(\Omega(m))$.

PROOF. Once we have obtained a formal expansion of $\pi_\sigma(g)\psi$ into a power series with variables the coordinate parameters of B and coefficients in $\mathcal{O}_\sigma(\Omega(m))$, Lemma 5.3 would imply that the expansion is indeed convergent. In view of (5.1), it suffices to consider $\pi_\sigma(g)\psi(\mathbf{u})$ for g in U^- , U_L^- and T (note that U^+ and U_L^+ are the conjugations of U^- and U_L^- by the long Weyl element).

Let $u \in U^-$, then

$$\pi_\sigma(u)\psi(\mathbf{u}) = \psi(u^{-1} \cdot \mathbf{u}).$$

Let $\mathcal{O}(\Omega(m))[[u]]$ denote the ring of formal power series $\varphi(u)$ in the coordinates u_α ($\alpha \in R_I^-$) with coefficients in $\mathcal{O}(\Omega(m))$, where $\varphi(u)$ is expressed as

$$\varphi(u) = \sum_{\underline{r} \in \mathbb{N}_0^{R_I^-}} a_{\underline{r}} \cdot \underline{u}^{\underline{r}}, \quad a_{\underline{r}} \in \mathcal{O}(\Omega(m)), \quad \underline{u}^{\underline{r}} := \prod_{\alpha \in R_I^-} u_\alpha^{r_\alpha}.$$

If the constant term a_0 is a unit in $\mathcal{O}(\Omega(m))$, then $\varphi(u)$ is invertible in $\mathcal{O}(\Omega(m))[[u]]$. Note that, for $\hat{g} \in \mathfrak{H}$, the constant term in the expansion of $f(\hat{g}, u^{-1} \cdot \mathbf{u})$ is $f(\hat{g}, \mathbf{u})$, and it is invertible in $\mathcal{O}(\Omega(m))$, so $f(\hat{g}, u^{-1} \cdot \mathbf{u})^{-1}$ belongs to $\mathcal{O}(\Omega(m))[[u]]$. Therefore, in view of the expansion form (4.10), the coordinates of $\psi(u^{-1} \cdot \mathbf{u})$ expand into a formal power series in u_α whose coefficients are series in $\mathcal{O}(\Omega(m))$, but it follows from Lemma 5.3 that the coefficients are indeed convergent series in $\mathcal{O}(\Omega(m))$ for $u \in B$. So each coordinate of $\psi(u^{-1} \cdot \mathbf{u})$ belongs to $\mathcal{O}(\Omega(m))[[u]]$.

For $l \in U_L^-$ or T ,

$$\pi_\sigma(l)\psi(\mathbf{u}) = \sigma(l)^{-1}\psi(l^{-1} \cdot \mathbf{u} \cdot l).$$

The arguments are similar.

Q.E.D.

COROLLARY 5.6. *Let $U_0^+ = U^+ \cap G_0$, then the power series expansion of*

$$f(j(u^+, \mathbf{u}))^{-1} = f(u^+ \cdot \mathbf{u})^{-1}, \quad u^+ \in U_0^+,$$

on U_0^+ converges in $\mathcal{O}(\Omega(m))$.

PROOF. Since f is an F -rational character on P^+ (Lemma 2.1 (1)), if we put $\sigma = f$ and $\psi \equiv 1$, then $(\pi_f(g^{-1})1)(\mathbf{u}) = f(j(g, \mathbf{u}))^{-1}$. Therefore our assertion follows from Lemma 5.5.

Q.E.D.

Now consider the dual representation π_σ^* of G on $\mathcal{O}_\sigma(\Omega)_b^* \cong \varinjlim_m \mathcal{O}(\Omega(m))_b^*$. The transition homomorphisms $\mathcal{O}_\sigma(\Omega(m))_b^* \rightarrow \mathcal{O}_\sigma(\Omega)_b^*$ are injective (see Proposition 4.8 (2)). Lemma 4.7 implies that, for any $g \in G_0$, $\pi_\sigma^*(g)$ maps $\mathcal{O}(\Omega(m))_b^*$ into $\mathcal{O}(\Omega(m'))_b^*$ via

$$\langle \psi, \pi_\sigma^*(g)\mu \rangle = \langle (\pi_\sigma(g^{-1})\psi)|_{\Omega(m)}, \mu \rangle, \quad \mu \in \mathcal{O}_\sigma(\Omega(m))_b^*, \psi \in \mathcal{O}_\sigma(\Omega(m'))_b^*.$$

We deduce from Lemma 5.5 that, for any $\mu \in \mathcal{O}_\sigma(\Omega(m))_b^*$, the orbit map

$$\begin{aligned} B^{-1} &\rightarrow \mathcal{O}_\sigma(\Omega(m'))_b^* \\ g &\mapsto \pi_\sigma^*(g)\mu \end{aligned}$$

is an $\mathcal{O}_\sigma(\Omega(m'))_b^*$ -valued analytic function. Therefore we obtain the following corollary.

COROLLARY 5.7. *$(\mathcal{O}_\sigma(\Omega)_b^*, \pi_\sigma^*)$ is locally analytic.*

6. Duality

In the following, we assume that K is spherically complete. Let (V, σ) be a d -dimensional K -rational representation of L . We choose a basis v_1, \dots, v_d of V and denote by v_1^*, \dots, v_d^* the corresponding dual basis of the dual space V^* . (V^*, σ^*) denotes the dual representation of (V, σ) .

6.1. The duality operator I_σ . For $\mathbf{u} \in \Omega$ and $v^* \in V^*$, let $\varphi_{\mathbf{u}, v^*}$ be the V^* -valued locally analytic function on \mathfrak{H} :

$$\varphi_{\mathbf{u}, v^*}(\hat{g}) := \sigma^*(j(\hat{g}, \mathbf{u}))v^*.$$

In view of (4.2), $\varphi_{\mathbf{u}, v^*}$ belongs to $C_{\sigma^*}^{\text{an}}(\mathfrak{H}, V^*)$. Let $B_{\sigma^*}^0(\mathfrak{H}, V^*)$ be the subspace of $C_{\sigma^*}^{\text{an}}(\mathfrak{H}, V^*)$ spanned by $\varphi_{\mathbf{u}, v^*}$, $B_{\sigma^*}(\mathfrak{H}, V^*)$ the closure of $B_{\sigma^*}^0(\mathfrak{H}, V^*)$. From (4.1), we see that $B_{\sigma^*}^0(\mathfrak{H}, V^*)$ and therefore $B_{\sigma^*}(\mathfrak{H}, V^*)$ are G -invariant.

For any continuous linear functional $\xi \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$, we define a V -valued function on Ω :

$$I_\sigma(\xi)(\mathbf{u}) := \sum_{k=1}^d \langle \varphi_{\mathbf{u}, v_k^*}, \xi \rangle v_k, \quad \mathbf{u} \in \Omega.$$

$I_\sigma(\xi)$ is independent of the choice of the basis $\{v_k\}_{k=1}^d$. Evidently, I_σ is injective.

LEMMA 6.1. I_σ is G -equivariant, that is,

$$I_\sigma(T_{\sigma^*}^*(g)\xi) = \pi_\sigma(g)I_\sigma(\xi),$$

for any $g \in G$.

PROOF.

$$\begin{aligned} I_\sigma(T_{\sigma^*}^*(g)\xi)(\mathbf{u}) &= \sum_{k=1}^d \langle \varphi_{\mathbf{u}, v_k^*}, T_{\sigma^*}^*(g)\xi \rangle v_k = \sum_{k=1}^d \langle T_{\sigma^*}(g^{-1})\varphi_{\mathbf{u}, v_k^*}, \xi \rangle v_k \\ &= \sum_{k=1}^d \langle \sigma^*(j(\cdot \cdot g^{-1}, \mathbf{u}))v_k^*, \xi \rangle v_k \\ &= \sigma(j(g^{-1}, \mathbf{u}))^{-1} \left(\sum_{k=1}^d \langle \sigma^*(j(\cdot, g^{-1} * \mathbf{u}))v_{k;g}^*, \xi \rangle v_{k;g} \right) \quad (\text{see (4.1)}) \\ &= (\pi_\sigma(g)I_\sigma(\xi))(\mathbf{u}), \end{aligned}$$

where $v_{k;g} = \sigma(j(g^{-1}, \mathbf{u}))v_k$, and similarly $v_{k;g}^* = \sigma^*(j(g^{-1}, \mathbf{u}))v_k^*$. $\{v_{k;g}\}_{k=1}^d$ and $\{v_{k;g}^*\}_{k=1}^d$ are dual to each other. Q.E.D.

PROPOSITION 6.2.

(1) For any continuous linear functional $\xi \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$, $I_\sigma(\xi)$ is a V -valued rigid analytic function on Ω .

(2) I_σ is a continuous homomorphism of G -representations from $(B_{\sigma^*}(\mathfrak{H}, V^*)^*)_b, T_{\sigma^*}^*)$ to $(\mathcal{O}_\sigma(\Omega), \pi_\sigma)$.

PROOF. *Step 1.* We denote by i the inclusion: $B_{\sigma^*}(\mathfrak{H}, V^*) \hookrightarrow C_{\sigma^*}^{\text{an}}(\mathfrak{H}, V^*)$, i^* its adjoint operator. Because of our assumption that K is spherically complete, the Hahn-Banach Theorem ([9] Corollary 9.4) implies that i^* is surjective. Since $C_{\sigma^*}^{\text{an}}(\mathfrak{H}, V^*)^*_b$ and $B_{\sigma^*}(\mathfrak{H}, V^*)^*_b$ are both Fréchet spaces (Corollary 3.6), the open mapping theorem ([9] Proposition 8.6) implies that i^* is open. Therefore the continuity of $I_\sigma \circ i^*$ implies that of I_σ . Consequently, (1) and (2) are equivalent to:

(1') $I_\sigma \circ i^*(\xi) \in \mathcal{O}_\sigma(\Omega)$ for any $\xi \in C_{\sigma^*}^{\text{an}}(\mathfrak{Y}, V^*)^*$;

(2') $I_\sigma \circ i^* : (C_{\sigma^*}^{\text{an}}(\mathfrak{Y}, V^*)^*, T_{\sigma^*}^*) \rightarrow (\mathcal{O}_\sigma(\Omega), \pi_\sigma)$ is a continuous homomorphism of G-representations.

Since G-equivariance is proved in Lemma 6.1, for (2') it remains to show the continuity of $I_\sigma \circ i^*$.

For convenience, we still denote $I_\sigma \circ i^*$ by I_σ .

Step 2. Let $\{\overline{\mathfrak{U}}_\kappa\}_\kappa$ be a finite disjoint open covering of $\overline{\mathfrak{Y}}$ satisfying:

1. $U_0^+ \in \{\overline{\mathfrak{U}}_\kappa\}_\kappa$ (note that the open subscheme $P^- \setminus C$ of $\overline{\mathfrak{Y}}$ is identified with U^+);
2. each $\overline{\mathfrak{U}}_\kappa$ is (right) translated into U_0^+ by some $g_\kappa \in G$.

Let \mathfrak{U}_κ be the preimage of $\overline{\mathfrak{U}}_\kappa$ under $\text{pr}_{\overline{\mathfrak{Y}}}^{\mathfrak{Y}}$.

For $\xi \in C_{\sigma^*}^{\text{an}}(\mathfrak{Y}, V^*)^*$, we write $I_\sigma(\xi)$ in integral:

$$\begin{aligned} I_\sigma(\xi)(\mathbf{u}) &= \sum_{k=1}^d \int_{\overline{\mathfrak{Y}}} \varphi_{\mathbf{u}; v_k^*} d\xi \cdot v_k = \sum_{k=1}^d \sum_{\kappa} \int_{\mathfrak{U}_\kappa} \varphi_{\mathbf{u}; v_k^*} d\xi \cdot v_k \\ &= \sum_{\kappa} \pi_\sigma(g_\kappa) \left(\sum_{k=1}^d \int_{\mathfrak{U}_\kappa \cdot g_\kappa} \varphi_{\mathbf{u}; v_{k; g_\kappa}^*} d(T_{\sigma^*}^*(g_\kappa^{-1})\xi) \cdot v_{k; g_\kappa} \right), \end{aligned}$$

where $v_{k; g_\kappa} = \sigma(j(g_\kappa^{-1}, \mathbf{u}))v_k$ is defined in the proof of Lemma 6.1. Therefore it suffices to consider

$$(6.1) \quad \sum_{k=1}^d \int_{\mathfrak{U}} \varphi_{\mathbf{u}; v_k^*} d\xi' \cdot v_k,$$

where \mathfrak{U} ranges on $\{\mathfrak{U}_\kappa \cdot g_\kappa\}_\kappa$ and ξ' is the image of ξ under $C_{\sigma^*}^{\text{an}}(\mathfrak{Y}, V^*)^* \rightarrow C_{\sigma^*}^{\text{an}}(\mathfrak{U}, V^*)^*$.

For the open subset $\overline{\mathfrak{U}} = \text{pr}_{\overline{\mathfrak{Y}}}^{\mathfrak{Y}}(\mathfrak{U})$ of U_0^+ , we have the isomorphism induced from a locally analytic section ι of $\text{pr}_{\mathfrak{U}}^{\mathfrak{U}}$ (see Lemma 3.5 (3)):

$$(6.2) \quad C_{\sigma^*}^{\text{an}}(\mathfrak{U}, V^*)^* \simeq C^{\text{an}}(\overline{\mathfrak{U}}, V^*)^*.$$

Then (6.1) is equal to

$$\bar{I}_{\sigma, \overline{\mathfrak{U}}}(\bar{\xi})(\mathbf{u}) := \sum_{k=1}^d \int_{\overline{\mathfrak{U}}} (\sigma^*(j(u^+, \mathbf{u}))v_k^*) d\bar{\xi}(u^+) \cdot v_k,$$

where $\bar{\xi}$ is the image of ξ' in $C^{\text{an}}(\overline{\mathfrak{U}}, V^*)^*$ via the isomorphism (6.2).

Therefore it suffices to prove that $\bar{I}_{\sigma, \overline{\mathfrak{U}}}(\bar{\xi})$ is rigid analytic on $\Omega(m)$, and that the map

$$\begin{aligned} C^{\text{an}}(\overline{\mathfrak{U}}, V^*)^* &\rightarrow \mathcal{O}_\sigma(\Omega(m)) \\ \bar{\xi} &\mapsto \bar{I}_{\sigma, \overline{\mathfrak{U}}}(\bar{\xi})|_{\Omega(m)} \end{aligned}$$

is continuous for all m .

Step 3. Since σ^* is K -rational, using the same arguments in the proof of Lemma 5.1 and applying Corollary 5.6, we obtain an expansion

$$\sigma^*(j(u^+, \mathbf{u}))v_k^* = \sum_{\ell=1}^d \left(\sum_{\underline{\ell} \in \mathbb{N}_0^{r_\ell^*}} a_{\underline{\ell}, k\ell}(\mathbf{u}) \cdot (\underline{u}^+)^{\underline{\ell}} \right) v_\ell^*,$$

with $a_{\underline{r},k\ell} \in \mathcal{O}(\Omega(m))$ such that

$$(6.3) \quad \lim_{|\underline{r}| \rightarrow \infty} \|a_{\underline{r},k\ell}\|_{\mathcal{O}(\Omega(m))} \cdot \|(\underline{u}^+)^{\underline{r}}\|_{C^{\text{an}}(\mathbb{U}_\sigma^+)} = 0,$$

and moreover, there is a constant $c' > 0$, depending only on m, σ and $\{v_k\}_{k=1}^d$, such that

$$(6.4) \quad \|a_{\underline{r},k\ell}\|_{\mathcal{O}(\Omega(m))} \cdot \|(\underline{u}^+)^{\underline{r}}\|_{C^{\text{an}}(\mathbb{U}_\sigma^+)} \leq c'.$$

Then

$$(6.5) \quad \bar{I}_{\sigma, \bar{\mathbb{U}}}(\bar{\xi})(\mathbf{u}) = \sum_{k=1}^d \left(\sum_{\ell=1}^d \sum_{\underline{r}} \int_{\bar{\mathbb{U}}} (\underline{u}^+)^{\underline{r}} \cdot v_\ell^* d\bar{\xi}(u^+) \cdot a_{\underline{r},k\ell}(\mathbf{u}) \right) v_k.$$

We have

$$(6.6) \quad \left| \int_{\bar{\mathbb{U}}} (\underline{u}^+)^{\underline{r}} \cdot v_\ell^* d\bar{\xi}(u^+) \right| \leq \|(\underline{u}^+)^{\underline{r}}\|_{C^{\text{an}}(\mathbb{U}_\sigma^+)} \cdot \|v_\ell^*\|_{V^*} \cdot \|\bar{\xi}\|_{C^{\text{an}}(\bar{\mathbb{U}}, V^*)^*}.$$

(6.3) and (6.6) imply that the expansion (6.5) of $\bar{I}_{\sigma, \bar{\mathbb{U}}}(\bar{\xi})$ converges in $\mathcal{O}_\sigma(\Omega(m))$.

(6.4) and (6.6) imply

$$\|\bar{I}_{\sigma, \bar{\mathbb{U}}}(\bar{\xi})\|_{\mathcal{O}_\sigma(\Omega(m))} \leq \max_{1 \leq k, \ell \leq d} c' \|v_\ell^*\|_{V^*} \|v_k\|_V \cdot \|\bar{\xi}\|_{C^{\text{an}}(\bar{\mathbb{U}}, V^*)^*},$$

and therefore the continuity follows.

Q.E.D.

6.2. The duality operator J_σ . Let $\mathcal{N}_\sigma(\Omega)$ denote the image of I_σ .

We consider J_σ , the adjoint operator of I_σ , which is an injective continuous linear operator from $\mathcal{N}_\sigma(\Omega)^*$ to $(B_{\sigma^*}(\mathfrak{H}, V^*)^*)^* \cong B_{\sigma^*}(\mathfrak{H}, V^*)$ ($B_{\sigma^*}(\mathfrak{H}, V^*)$ is reflexive according to Corollary 3.6).

For any $\mu \in \mathcal{N}_\sigma(\Omega)^*$ and $\xi \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$, we have

$$(6.7) \quad \langle J_\sigma(\mu), \xi \rangle = \langle I_\sigma(\xi), \mu \rangle.$$

For $\hat{g} \in \mathfrak{H}$ and $v \in V$, we define the Dirac distribution $\xi_{\hat{g}, v} \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$ as follows:

$$\langle \varphi, \xi_{\hat{g}, v} \rangle = \langle v, \varphi(\hat{g}) \rangle_V, \quad \varphi \in B_{\sigma^*}(\mathfrak{H}, V^*),$$

and a V -valued rigid analytic function $\psi_{\hat{g}, v}$ on Ω :

$$\psi_{\hat{g}, v}(\mathbf{u}) := \sigma(j(\hat{g}, \mathbf{u}))^{-1} v.$$

LEMMA 6.3.

$$I_\sigma(\xi_{\hat{g}, v}) = \psi_{\hat{g}, v}.$$

PROOF. This is straightforward from definitions.

$$\begin{aligned} I_\sigma(\xi_{\hat{g}, v})(\mathbf{u}) &= \sum_{k=1}^d \langle \varphi_{\mathbf{u}, v_k^*}, \xi_{\hat{g}, v} \rangle v_k = \sum_{k=1}^d \langle v, \varphi_{\mathbf{u}, v_k^*}(\hat{g}) \rangle_V v_k = \sum_{k=1}^d \langle v, \sigma^*(j(\hat{g}, \mathbf{u})) v_k^* \rangle_V v_k \\ &= \sum_{k=1}^d \langle \sigma(j(\hat{g}, \mathbf{u})) v, v_k^* \rangle_V v_k = \sigma(j(\hat{g}, \mathbf{u})) v = \psi_{\hat{g}, v}(\mathbf{u}) \end{aligned}$$

Q.E.D.

Then we obtain a formula for J_σ .

PROPOSITION 6.4. *For any continuous linear functional $\mu \in \mathcal{N}_\sigma(\Omega)^*$, we have*

$$(6.8) \quad J_\sigma(\mu)(\hat{g}) = \sum_{k=1}^d \langle \psi_{\hat{g}, v_k}, \mu \rangle v_k^*.$$

PROOF. This is straightforward from Lemma 6.3 and (6.7). Indeed,

$$\begin{aligned} \sum_{k=1}^r \langle \psi_{\hat{g}, v_k}, \mu \rangle v_k^* &= \sum_{k=1}^r \langle I_\sigma(\xi_{\hat{g}, v_k}), \mu \rangle v_k^* = \sum_{k=1}^r \langle J_\sigma(\mu), \xi_{\hat{g}, v_k} \rangle v_k^* \\ &= \sum_{k=1}^r \langle v_k, J_\sigma(\mu)(\hat{g}) \rangle_V \cdot v_k^* = J_\sigma(\mu)(\hat{g}). \end{aligned}$$

Q.E.D.

6.3. The image of I_σ . Let $\mathcal{N}_\sigma^0(\Omega)$ denote the subspace of $\mathcal{O}_\sigma(\Omega)$ spanned by $\psi_{\hat{g}, v}$ for all $\hat{g} \in \mathfrak{H}$ and $v \in V$. Then it follows from (4.1) that $\mathcal{N}_\sigma^0(\Omega)$ is G -invariant, and Lemma 6.3 implies $\mathcal{N}_\sigma^0(\Omega) \subset \mathcal{N}_\sigma(\Omega)$.

From (6.8), we see that J_σ factors through $\mathcal{N}_\sigma^0(\Omega)^*$, and (6.8) defines an injective map from $\mathcal{N}_\sigma^0(\Omega)_b^*$ into $B_{\sigma^*}(\mathfrak{H}, V^*)$. Since J_σ is injective and the natural map $\mathcal{N}_\sigma(\Omega)_b^* \rightarrow \mathcal{N}_\sigma^0(\Omega)_b^*$ is surjective (the Hahn-Banach Theorem), $\mathcal{N}_\sigma^0(\Omega)_b^* = \mathcal{N}_\sigma(\Omega)_b^*$. Therefore the Hahn-Banach Theorem implies the following lemma.

LEMMA 6.5. *$\mathcal{N}_\sigma^0(\Omega)$ is dense in $\mathcal{N}_\sigma(\Omega)$.*

THEOREM 6.6.

- (1) I_σ is an isomorphism from $B_{\sigma^*}(\mathfrak{H}, V^*)_b^*$ to $\mathcal{N}_\sigma(\Omega)$.
- (2) $\mathcal{N}_\sigma(\Omega)$ is the closure of $\mathcal{N}_\sigma^0(\Omega)$ in $\mathcal{O}_\sigma(\Omega)$.

PROOF. Let ι be a locally analytic section of $\text{pr}_{\mathfrak{H}}^{\mathfrak{H}}$, and denote $\mathfrak{K} = \iota(\overline{\mathfrak{H}})$.

1. Let $\mathcal{N}_\sigma^0(\Omega(m))$ be the image of $\mathcal{N}_\sigma^0(\Omega)$ in $\mathcal{O}_\sigma(\Omega(m))$.

Since $\psi_{\hat{g}, v_k} = \pi_\sigma(g^{-1})v_k$, we see that the map

$$\begin{aligned} \mathfrak{H} &\rightarrow \mathcal{O}_\sigma(\Omega(m)) \\ \hat{g} &\mapsto \psi_{\hat{g}, v_k}, \end{aligned}$$

is locally analytic (Lemma 5.5). Since \mathfrak{K} is compact, $\rho_m = \min_{1 \leq k \leq d} \min_{\hat{g} \in \mathfrak{K}} \|\psi_{\hat{g}, v_k}\|_{\mathcal{O}_\sigma(\Omega(m))}$ is positive.

Let \mathcal{L} denote the lattice $\sum_{k=1}^d \sum_{\hat{g} \in \mathfrak{K}} \mathfrak{o}_K \cdot \psi_{\hat{g}, v_k}$ in $\mathcal{N}_\sigma^0(\Omega)$. Then, for each m , the image of \mathcal{L} in $\mathcal{N}_\sigma^0(\Omega(m))$ contains the ball of radius ρ_m centered at zero, and therefore the interior of \mathcal{L} is a nontrivial open lattice.

2. According to Lemma 3.5, ι induces an isomorphism ι° between $C_{\sigma^*}^{\text{an}}(\mathfrak{H}, V^*)$ and $C^{\text{an}}(\overline{\mathfrak{H}}, V^*)$, and hence an isomorphism between $B_{\sigma^*}(\mathfrak{H}, V^*)$ and its image, denoted by $B(\overline{\mathfrak{H}}, V^*)$.

Let \mathcal{I} be any (finite) disjoint open chart covering $(\overline{\mathfrak{U}_k})_K$ of $\overline{\mathfrak{H}}$. We recall that $C^{\text{an}}(\overline{\mathfrak{H}}, V^*)$ is defined to be the inductive limit, indexed with all the \mathcal{I} , of the K -Banach algebras $E_{\mathcal{I}}(\overline{\mathfrak{H}}, V^*) = \prod_K \mathcal{O}(\overline{\mathfrak{U}_k}, V^*)$, where $\mathcal{O}(\overline{\mathfrak{U}_k}, V^*)$ denotes the space of K -analytic functions

on $\overline{\mathfrak{U}}_\kappa$ (cf. [2] 2.1.10 and [11] §2). The inductive limit structure is naturally induced onto $B(\overline{\mathfrak{S}}, V^*)$, say $B(\overline{\mathfrak{S}}, V^*) = \varinjlim_I E_I(\overline{\mathfrak{S}}, V^*)$. Moreover, the strong dual space $B(\overline{\mathfrak{S}}, V^*)_b^*$ is the projective limit of $E_I(\overline{\mathfrak{S}}, V^*)_b^*$.

3. Consider

$$\begin{aligned} (\iota^{\circ-1})^* \circ I_\sigma^{-1}|_{\mathcal{N}_\sigma^0(\Omega)} : \mathcal{N}_\sigma^0(\Omega) &\rightarrow B(\overline{\mathfrak{S}}, V^*)_b^* \\ \psi_{\hat{g}, v} &\mapsto (\iota^{\circ-1})^*(\xi_{\hat{g}, v}). \end{aligned}$$

Let $\hat{g} \in \mathfrak{R}$.

$$\begin{aligned} \|(\iota^{\circ-1})^*(\xi_{\hat{g}, v})\|_{E_I(\overline{\mathfrak{S}}, V^*)_b^*} &= \max_{\overline{\varphi} \in E_I(\overline{\mathfrak{S}}, V^*)} \frac{\langle \overline{\varphi}, (\iota^{\circ-1})^*(\xi_{\hat{g}, v}) \rangle}{\|\overline{\varphi}\|_{E_I(\overline{\mathfrak{S}}, V^*)}} \\ &= \max_{\varphi \in \iota^{\circ-1}(E_I(\overline{\mathfrak{S}}, V^*))} \frac{\langle \varphi, \xi_{\hat{g}, v} \rangle}{\|\iota^{\circ}(\varphi)\|_{E_I(\overline{\mathfrak{S}}, V^*)}} \\ &= \max_{\varphi \in \iota^{\circ-1}(E_I(\overline{\mathfrak{S}}, V^*))} \frac{\langle v, \varphi(\hat{g}) \rangle_V}{\max_{\hat{g}' \in \mathfrak{R}} \|\varphi(\hat{g}')\|_{V^*}} \\ &\leq \|v\|_V. \end{aligned}$$

Therefore the image of \mathcal{L} under $(\iota^{\circ-1})^* \circ I_\sigma^{-1}|_{\mathcal{N}_\sigma^0(\Omega)}$ in $B(\overline{\mathfrak{S}}, V^*)_b^*$ is bounded, since its image in $E_I(\overline{\mathfrak{S}}, V^*)_b^*$ are all norm-bounded by $\max_{1 \leq k \leq d} \|v_k\|_V$. Because $\mathcal{N}_\sigma^0(\Omega)$ is metrizable, it is bornological ([9] Proposition 6.14), and therefore $I_\sigma^{-1}|_{\mathcal{N}_\sigma^0(\Omega)}$ is continuous ([9] Proposition 6.13). Therefore I_σ induces an isomorphism between $I_\sigma^{-1}(\mathcal{N}_\sigma^0(\Omega))$ and $\mathcal{N}_\sigma^0(\Omega)$, and consequently I_σ induces an isomorphism between their completions, which, in view of Lemma 6.5, must be $B_{\sigma^*}(\mathfrak{S}, V^*)_b^*$ and $\mathcal{N}_\sigma(\Omega)$ respectively. Q.E.D.

COROLLARY 6.7. *J_σ is an isomorphism of G -representations from $(\mathcal{N}_\sigma(\Omega)_b^*, \pi_\sigma^*)$ to $(B_{\sigma^*}(\mathfrak{S}, V^*), T_{\sigma^*})$.*

7. Concluding remarks

In [7] §3 we briefly reviewed Morita's theory of $\mathrm{SL}(2, F)$ and discussed the relation between I_σ and Morita's duality and Casselman's operator for

$$\sigma_s \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = z^s$$

with s a positive integer (for s non-positive, I_σ is an isomorphism between two $(-s+1)$ -dimensional G -representations, which is of less interest).

To illustrate this connection, we consider the special case $s = 2$. $\mathcal{O}_{\sigma_2}(\Omega)$ is canonically isomorphic to the space $\Omega^1(\Omega)$ of holomorphic 1-forms on the upper half plane $\Omega \cong K - F = \mathbb{P}^1(K) - \mathbb{P}^1(F)$ via $\psi(\mathbf{u}) \mapsto \psi(\mathbf{u})d\mathbf{u}$. It may be shown that $\mathcal{N}_{\sigma_2}(\Omega)$ corresponds to the subspace of $\Omega^1(\Omega)$ with zero residue at each point of $\mathbb{P}^1(F)$ (see [4] and [7]). On the other hand, $C_{\sigma_0}^{\mathrm{an}}(\mathfrak{S}) \cong C^{\mathrm{an}}(\overline{\mathfrak{S}})$ with $\overline{\mathfrak{S}} \cong \mathbb{P}^1(F)$, and we denote $D_0 = C^{\mathrm{an}}(\mathbb{P}^1(F))$. D_0 has two closed G -invariant subspaces, the spaces P_0 and P_0^{loc} consisting of constants and locally

constant functions on $\mathbb{P}^1(F)$ respectively. The classical Morita's duality is established via residues. More precisely, for each $\psi \in \mathcal{O}_{\sigma_2}(\Omega)$, define a linear functional $M_2(\psi)$ of D_0 by

$$\langle \varphi, M_2(\psi) \rangle = \text{the sum of residues of the 1-form } \varphi(u)\psi(u) du \text{ on } \mathbb{P}^1(F).$$

Morita's duality M_2 induces G-isomorphisms $\mathcal{O}_{\sigma_2}(\Omega) \cong (D_0/P_0)_b^*$ and $\mathcal{N}_{\sigma_2}(\Omega) \cong (D_0/P_0^{\text{loc}})_b^*$. Moreover, Casselman's intertwining operator

$$S_0 : \varphi \mapsto d\varphi$$

induces a G-isomorphism between D_0/P_0^{loc} and the space D_{-2} of locally analytic 1-forms on $\mathbb{P}^1(F)$, a space that is isomorphic to $C_{\sigma_{-2}}^{\text{an}}(\mathfrak{H})$ (see [5] and [7]). The connection between our duality operator I_{σ_2} and Morita's duality M_2 was found in [7] Theorem 3.6 as the following commutative diagram

$$\begin{array}{ccc} \mathcal{N}_{\sigma_2}(\Omega) & \xrightarrow{M_2} & (D_0/P_0^{\text{loc}})_b^* \\ \uparrow I_{\sigma_2} & & \uparrow S_0^* \\ B_{\sigma_{-2}}(\mathfrak{H})_b^* = C_{\sigma_{-2}}^{\text{an}}(\mathfrak{H})_b^* & \cong & (D_{-2})_b^* \end{array}$$

A generalization of Morita's duality seems quite hard in view of its analytic construction via residues. The first step towards this would be finding other closed subrepresentations of $(C_{\sigma}^{\text{an}}(\mathfrak{H}, V), T_{\sigma})$ and $(\mathcal{O}_{\sigma}(\Omega), \pi_{\sigma})$. This work is done completely for $\text{SL}(2, F)$ in Morita and Murase's [4], [5] and [6], where the complete classifications of the sub-quotient spaces of holomorphic discrete series and the principal series are conjectured and claimed (Morita attempted to prove this, but his proof contained a serious gap).

A further question is on the irreducibility. For this, we conjecture that $(\mathcal{N}_{\sigma}(\Omega), \pi_{\sigma})$ and $(B_{\sigma^*}(\mathfrak{H}, V^*), T_{\sigma^*})$ are topologically irreducible G-representations if σ is irreducible. For $\text{SL}(2, F)$, this conjecture was claimed in [6] Theorem 1 (i) and a proof for $F = \mathbb{Q}_p$ was given by Schneider and Teitelbaum in [11].

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